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$D = 3$ real quantum conformal algebra

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Abstract. Using the equivalence of $o(3, 2)$ with $D = 3$ conformal algebras we describe the $D = 3$ quantum conformal algebra by the q-deformed Cartan–Weyl basis of a real Hopf algebra $U_q(o(3, 2))$ with $|q| = 1$. The universal R-matrix written in terms of $D = 3$ conformal generators is investigated. The Hopf subalgebras of $U_q(o(3, 2))$ and their quasitriangularity are discussed. The analogies with $D = 4$ quantum conformal algebra are pointed out.

1. Introduction

Recently the formalism of quantum groups [1–3] has been applied to the spacetime symmetries [4–14]; in particular the most important case of $D = 4$ Poincaré algebra was considered in [4–7, 9, 10, 12–14]. In this paper we would like to consider the quantum deformations of $D = 3$ spacetime symmetries. Let us recall that in $D = 3$ the following algebras describe spacetime symmetries:

(a) $D = 3$ Lorentz algebra—$o(2, 1) \simeq su(1, 1)$,
(b) $D = 3$ Poincaré algebra,
(c) $D = 3$ anti-de-Sitter algebra—$o(2, 2) \simeq o(2, 1) \oplus o(2, 1)$,
(d) $D = 3$ de-Sitter algebra—$o(3, 1) \simeq sl(2, \mathbb{C})$,
(e) $D = 3$ conformal algebra—$o(3, 2) \simeq Sp(4, \mathbb{R})$.

The first three algebras are described by Lie algebras of rank one. The quantum deformations of Lie algebras of rank one have been thoroughly studied. The quantum deformation of $D = 3$ Poincaré algebra has been obtained in [15] by considering the infinite de-Sitter radius limit of the $D = 3$ anti-de-Sitter algebra described by the sum of two rank one algebras. The most interesting case is given by the q-deformed $D = 3$ conformal algebra. It appears that in such a case one can use the results obtained for the quantum deformed real form of $B_2 \simeq C_2$ Lie algebra, i.e. $U_q(Sp(4, \mathbb{C}))$, see e.g. [4]. Using a properly chosen real form one can deform the $D = 3$ conformal algebra in such a way that its Lorentz subalgebra is described by a real Hopf subalgabra.

In this paper we rewrite the Cartan–Weyl basis of $U_q(o(3, 2))$ in terms of $D = 3$ quantum conformal algebra generators. Further we write down (in terms of these generators) the universal R-matrix and study its Hopf subalgabras. Finally we compare some results obtained for quantum $D = 4$ conformal algebra with our $D = 3$ case. It appears that contrary to the case $D = 4$ (see [7, 8]) if $D = 3$ one can obtain the standard real Hopf algebra structure for quantum Weyl and quantum Lorentz subalgebras.

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2. Cartan–Weyl basis of $U_q(Sp(4,\mathbb{C}))$

From the Dynkin diagram of $Sp(4,\mathbb{C})$

$$\alpha_1 \quad \circ\!\Longrightarrow\!\bullet \quad \alpha_2$$

we can deduce its symmetrized Cartan matrix ($\alpha_{ij} = \langle\alpha_i, \alpha_j\rangle$, $i = 1, 2$)

$$\alpha = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then the algebra $Sp(4,\mathbb{C})$ is known to be generated by elements $(h_i, e_i, e_{-i})_{i=1,2}$ satisfying the following conditions:

$$\begin{aligned}
&[e_i, e_{-j}] = \delta_{ij} h_i \\
&[h_i, e_{\pm j}] = \pm\alpha_{ij} e_j \\
&[h_i, h_j] = 0 \\
&[e_{\pm 1}, [e_{\pm 1}, [e_{\pm 1}, e_{\pm 2}]]] = 0 \\
&[e_{\pm 2}, [e_{\pm 2}, e_{\pm 1}]] = 0
\end{aligned} \tag{1}$$

and is spanned by $(h_i, e_i, e_{-i})_{i=1,2}$, $e_{\pm 3} = [e_{\pm 1}, e_{\pm 2}]$, $e_{\pm 4} = [e_{\pm 1}, e_{\pm 3}]$.

The Drinfeld–Jimbo q-deformation of its enveloping algebra $U_q(Sp(4,\mathbb{C}))$ is defined to be generated by $(h_i, e_i, e_{-i})_{i=1,2}$ satisfying modified relations

$$\begin{aligned}
&[e_i, e_{-j}] = \delta_{ij} [h_i]_q \\
&[h_i, e_{\pm j}] = \pm\alpha_{ij} e_j \\
&[h_i, h_j] = 0 \\
&[e_{\pm 1}, [e_{\pm 1}, [e_{\pm 1}, e_{\pm 2}]_{q^{\pm 1}}]_{q^{\pm 1}}]_{q^{\pm 1}} = 0 \\
&[e_{\pm 2}, [e_{\pm 2}, e_{\pm 1}]_{q^{\mp 1}}]_{q^{\mp 1}} = 0
\end{aligned} \tag{2}$$

where

$$[x]_q := \frac{(q^x - q^{-x})}{(q - q^{-1})} = \frac{\sinh \hbar x}{\sinh \hbar} \qquad q = \exp \hbar \tag{3}$$

$$[e_m, e_n]_q := e_m e_n - q^{-\alpha_{mn}} e_n e_m. \tag{4}$$

Here e_m, e_n are just any elements of the mth and nth root spaces in $U(Sp(4,\mathbb{C}))$ and α_{mn} is the scalar product of the corresponding roots. However, in what follows we will use the q-bracket notation only in the meaning

$$[A, B]_q = ABq^{-1/2} - q^{1/2} BA. \tag{5}$$

$A := U_q(Sp(4,\mathbb{C}))$ may be given a Hopf algebra structure by defining the coproduct $\Delta : A \to A \otimes A$ as an algebra homomorphism acting on the generators in the following way:

$$\begin{aligned}
&\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \\
&\Delta(e_{\pm i}) = e_{\pm i} \otimes q^{h_i/2} + q^{-h_i/2} \otimes e_{\pm i}
\end{aligned} \tag{6}$$

and the antipode $S : A \to A$ as the algebra antihomomorphism taking the following values on the generators:

$$\begin{aligned}
&S(h_i) = -h_i \\
&S(e_{\pm i}) = -q^{\pm d_i/2} e_{\pm i}
\end{aligned} \tag{7}$$

where $d_i = \alpha_{ii}$ $\quad (i = 1, 2)$.

The Cartan–Weyl basis of $U_q(Sp(4,\mathbb{C})$ consists of generators $(h_i, e_i, e_{-i})_{i=1,2}$ and the elements $E_{\pm3}$, $E_{\pm4}$ defined below

$$\begin{aligned} E_3 &= [e_1, e_2]_q \qquad & E_{-3} &= [e_{-2}, e_{-1}]_{q^{-1}} \\ E_4 &= [e_1, E_3] \qquad & E_{-4} &= [E_{-3}, e_{-1}]. \end{aligned} \tag{8}$$

3. Real Hopf algebras: $U_q(o(3,2))$ and quantum $D = 3$ conformal algebra

In order to obtain a $U_q(o(3,2))$ a real form of $U_q(Sp(4,\mathbb{C}))$ must be chosen. In the case of $U_q(g)$ algebras real forms are identified with pairs $(U_q(g), ^\star)$ where $^\bullet$ is a Hopf algebra involution of $U_q(g)$ [16]. Various types of involutions of $U_q(Sp(4,\mathbb{C}))$ have been studied [17] and denoted with $^+, ^*, ^\otimes, ^\oplus$ depending on whether they are automorphisms or antiautomorphisms of algebra and coalgebra structures of $U_q(g)$:

$$+ \quad : \quad (ab)^+ = b^+a^+ \quad (\Delta(a))^+ = \Delta(a^+) \tag{9}$$

$$* \quad : \quad (ab)^* = a^*b^* \quad (\Delta(a))^* = \tau \circ \Delta(a^*) \tag{10}$$

$$\oplus \quad : \quad (ab)^\oplus = b^\oplus a^\oplus \quad (\Delta(a))^\oplus = \tau \circ \Delta(a^\oplus) \tag{11}$$

$$\otimes \quad : \quad (ab)^\otimes = a^\otimes b^\otimes \quad (\Delta(a))^\otimes = \Delta(a^\otimes) \tag{12}$$

where $\tau(a \otimes b) = b \otimes a$.

The list of all standard and non-standard real forms of $U_q(o(3-k,k))$ $(k = 0, 1, 2)$ is given in [17]. In this paper we shall consider only those standard real forms of $U_q(o(3,2))$ which contain $U_q(o(2,1))$ as their real Hopf subalgebra. The following eight real forms can be obtained [17]:

(i) $|q| = 1 \quad (\Delta_\pm \to \Delta_\pm; \lambda^2 = 1, \epsilon^2 = 1)$

$$\begin{aligned} k_i^+ &= k_i \qquad & h_i^+ &= -h_i \\ e_{\pm1}^+ &= \lambda e_{\pm1} \qquad & e_{\pm2}^+ &= \epsilon e_{\pm2} \\ E_{\pm3}^+ &= -\lambda\epsilon E_{\pm3} \qquad & E_{\pm4}^+ &= \epsilon E_{\pm4} \end{aligned} \tag{13}$$

(ii) q real $\quad (\Delta_\pm \to \Delta_\mp; \lambda^2 = 1, \epsilon^2 = 1)$

$$\begin{aligned} k_i^+ &= k_i \qquad & h_i^+ &= h_i \\ e_{\pm1}^+ &= \lambda e_{\mp1} \qquad & e_{\pm2}^+ &= \epsilon e_{\mp2} \\ E_{\pm3}^+ &= -\lambda\epsilon \tilde{E}_{\mp3} \qquad & E_{\pm4}^+ &= \epsilon \tilde{E}_{\mp4}. \end{aligned} \tag{14}$$

It turns out that in case (ii) the reality conditions relate the elements $E_{\pm3}$, $E_{\pm4}$ of the Cartan–Weyl basis to the antipode extended generators $\tilde{E}_{\pm3}$, $\tilde{E}_{\pm4}$:

$$\begin{aligned} \tilde{E}_3 &= [e_2, e_1]_q \qquad & \tilde{E}_{-3} &= [e_{-1}, e_{-2}]_{q^{-1}} \\ \tilde{E}_4 &= [\tilde{E}_3, e_1] \qquad & \tilde{E}_{-4} &= [e_{-1}, \tilde{E}_{-3}]. \end{aligned} \tag{15}$$

Therefore, we will restrict ourselves to case (i), with involution $^+$ acting as an inner morphism in the Cartan–Weyl basis. We will also choose the values $\lambda = \epsilon = -1$.

Then the $o(3,2)$ rotation generators $(M_{AB} = -M_{BA})$,

$$\begin{gathered} M_{12} = h_1 \qquad M_{04} = h_3 \qquad M_{23} = \frac{1}{\sqrt{2}}(e_1 + e_{-1}) \\ M_{03} = \frac{1}{\sqrt{2}}(E_3 + E_{-3}) \qquad M_{31} = \frac{1}{\sqrt{2}}(e_1 - e_{-1}) \qquad M_{34} = \frac{1}{\sqrt{2}}(E_3 - E_{-3}) \\ M_{24} = -\tfrac{1}{2}(e_2 + e_{-2} + E_4 + E_{-4}) \qquad M_{14} = -\tfrac{1}{2}(e_2 - e_{-2} - E_4 + E_{-4}) \\ M_{02} = -\tfrac{1}{2}(e_2 - e_{-2} + E_4 - E_{-4}) \qquad M_{01} = -\tfrac{1}{2}(e_2 + e_{-2} - E_4 - E_{-4}) \end{gathered} \tag{16}$$

satisfy the 'reality condition'

$$M_{AB}^{+} = -M_{AB} \tag{17}$$

and in the limit $q \to 1$ satisfy the standard relations

$$[M_{AB}, M_{CD}] = g_{BC}M_{AD} + g_{AD}M_{BC} - g_{AC}M_{BD} - g_{BD}M_{AC} \tag{18}$$

where $A, B, C, D \in \{0, 1, 2, 3, 4\}$ and $g = \mathrm{diag}(- + - + +)$. Thus we may treat the pair $(U_q(Sp(4, \mathbb{C})), ^{+})$ with generators satisfying (17) as a real Hopf algebra $U_q(o(3, 2))$ [16].

Now we interpret the subalgebra generated by (M_{12}, M_{23}, M_{31}) as the Lorentz subalgebra of $D = 3$ conformal algebra and define other conformal generators ($i = 1, 2, 3$):

$$K_i = (M_{0i} + M_{4i})/\sqrt{2} \qquad P_i = (M_{0i} - M_{4i})/\sqrt{2} \qquad D = M_{04} \tag{19}$$

which in the limit $q \to 1$ satisfy

$$\begin{array}{lll} [D, M_{ij}] = 0 & [K_i, K_j] = 0 & \\ [D, K_i] = K_i & [P_i, P_j] = 0 & [M_{ij}, K_k] = g_{jk}K_i - g_{ik}K_j \\ [D, P_i] = -P_i & [K_i, P_j] = g_{ij}D & [M_{ij}, P_k] = g_{jk}P_i - g_{ik}P_j. \end{array} \tag{20}$$

To obtain a greater clearness of subsequent formulas we define

$$\begin{array}{ll} M_0 = M_{12} & M_{\pm} = (M_{23} \pm M_{31})/\sqrt{2} \\ K_0 = K_3 & K_{\pm} = (K_2 \mp K_1)/\sqrt{2} \\ P_0 = P_3 & P_{\pm} = (P_2 \mp P_1)/\sqrt{2}. \end{array} \tag{21}$$

In terms of the generators M, P, K, D the $U_q(o(3, 2))$ commutation relations take the following form:

$$\begin{array}{ll} [M_0, M_{\pm}] = \pm M_{\pm} & [M_+, M_-] = [M_0]_q \\ [K_0, K_{\pm}]_{q^{\pm 1}} = 0 & [K_+, K_-] = (q^{1/2} - q^{-1/2})K_0^2 \\ [P_0, P_{\pm}]_{q^{\mp 1}} = 0 & [P_+, P_-] = (q^{1/2} - q^{-1/2})P_0^2. \end{array} \tag{22}$$

Sector (M, P):

$$\begin{array}{lll} [M_+, P_+] = 0 & [M_0, P_+] = +P_+ & [M_-, P_+]_q = P_0 \\ [M_+, P_0] = -q^{-M_0+\frac{1}{2}}P_+ & [M_0, P_0] = 0, & [M_-, P_0] = P_- \\ [M_+, P_-] = -q^{M_0}P_0 & [M_0, P_-] = -P_- & [M_-, P_-]_{q^{-1}} = 0. \end{array} \tag{23}$$

Sector (M, K):

$$\begin{array}{lll} [M_+, K_+]_{q^{-1}} = 0 & [M_0, K_+] = K_+ & [M_-, K_+] = K_0 q^{M_0} \\ [M_+, K_0] = K_+ & [M_0, K_0] = 0 & [M_-, K_0] = K_- q^{M_0 - \frac{1}{2}} \\ [M_+, K_-]_q = K_0 & [M_0, K_-] = -K_- & [M_-, K_-] = 0. \end{array} \tag{24}$$

Sector (K, P):

$$\begin{array}{l} [K_+, P_+] = (q^{1/2} - q^{-1/2})q^{M_0 - D - 1}M_+^2 \\ [K_+, P_0] = q^{-D}M_+ \qquad [K_+, P_-] = -[M_0 + D]_q \\ [K_0, P_+] = -q^{M_0 - D - \frac{1}{2}}M_+ \qquad [K_0, P_0] = [D]_q \qquad [K_0, P_-] = -M_- q^D \\ [K_-, P_+] = [M_0 - D]_q \qquad [K_-, P_0] = M_- q^{D - M_0 + \frac{1}{2}} \\ [K_-, P_-] = -(q^{1/2} - q^{-1/2})M_-^2 q^{D - M_0 + 1}. \end{array} \tag{25}$$

Sector (D, all other generators):

$$\left.\begin{aligned} [D, M_i] &= 0 \\ [D, P_i] &= -P_i \\ [D, K_i] &= K_i \end{aligned}\right\} \quad \text{for } i = (+, -, 0). \tag{26}$$

The coproduct relations read:

$$\begin{aligned}
&(M_0) = M_0 \otimes 1 + 1 \otimes M_0 \\
&\Delta(M_\pm) = M_\pm \otimes q^{M_0/2} + q^{-M_0/2} \otimes M_\pm \\
&\Delta(D) = D \otimes 1 + 1 \otimes D \\
&\Delta(P_+) = P_+ \otimes q^{(D-M_0)/2} + q^{(M_0-D)/2} \otimes P_+ \\
&\Delta(P_0) = P_0 \otimes q^{D/2} + q^{-D/2} \otimes P_0 \tfrac{1}{2}(q - q^{-1}) q^{-(M_0-\frac{1}{2})/2} P_+ \otimes M_- q^{(M_0+\frac{1}{2})/2} \\
&\Delta(P_-) = P_- \otimes q^{(M_0+D)/2} + q^{-(M_0+D)/2} \otimes P_+ \\
&\qquad + (q - q^{-1})[(q^{1/2} - q^{-1/2}) q^{-M_0+\frac{1}{2}} P_+ \otimes M_-^2 q^{(D-M_0+1)/2} \\
&\qquad + q^{-M_0/2} P_0 \otimes M_- q^{D/2}] \\
&\Delta(K_+) = K_+ \otimes q^{(M_0+D)/2} + q^{-(M_0+D)/2} \otimes K_+ \\
&\qquad + (q - q^{-1})[(q^{1/2} - q^{-1/2}) q^{(-M_0-D-1)/2} M_+^2 \otimes K_- q^{(D-M_0-1)/2} \\
&\qquad - q^{-D/2} M_+ \otimes K_0 q^{M_0/2}] \\
&\Delta(K_0) = K_0 \otimes q^{D/2} + q^{-D/2} \otimes K_0 (q - q^{-1}) q^{(-D+M_0-\frac{1}{2})/2} M_+ \otimes K_- q^{(M_0-\frac{1}{2})/2} \\
&\Delta(K_-) = K_- \otimes q^{(D-M_0)/2} + q^{(M_0-D)/2} \otimes K_-.
\end{aligned} \tag{27}$$

The values of the antipode are:

$$\begin{aligned}
&S(M_0) = -M_0 \qquad S(M_\pm) = -q^{\pm\frac{1}{2}} M_\pm \qquad S(D) = -D \\
&S(P_+) = -q^{-1} P_+ \qquad S(P_0) = -q^{-1/2} P_0 + (1 - q^{-2}) P_+ M_- \\
&S(P_-) = -q^{-1} P_- + (1 - q^{-2})[(q^{-1/2} - q^{1/2}) P_+ M_-^2 - P_0 M_-] \\
&S(K_+) = -q K_+ + (1 - q^2)[(q^{1/2} - q^{-1/2}) M_+^2 K_- + M_+ K_0] \\
&S(K_0) = -q^{1/2} K_0 + (1 - q^2) M_+ K_- \\
&S(K_-) = -q K_-.
\end{aligned} \tag{28}$$

4. Universal R-matrix

A Hopf algebra A is called quasitriangular Hopf algebra if there exists an element R in an appropriate completion of $A \otimes A$ which satisfies

$$R\Delta(x) = \Delta'(x) R \qquad \text{for all } x \in A \tag{29}$$

$$(\Delta \otimes \mathrm{id}) R = R_{13} R_{12} \qquad (\mathrm{id} \otimes \Delta) R = R_{12} R_{13} \tag{30}$$

where $\Delta' = \tau \circ \Delta$ is the opposite coproduct and

$$R_{23} = 1 \otimes R \qquad R_{12} = R \otimes 1 \qquad R_{13} = (\tau \otimes \mathrm{id}) \circ R_{23}. \tag{31}$$

The universal R-matrix is known to exist for all $U_q(g)$ Hopf algebras [18, 19]. The formula for R has been given in [19] but for a different coproduct namely:

$$\begin{aligned}
&\Delta_T(E_i) = E_i \otimes 1 + q^{h_i} \otimes E_i \\
&\Delta_T(E_{-i}) = E_{-i} \otimes q^{h_i} + 1 \otimes E_{-i} \\
&\Delta_T(h_i) = h_i \otimes 1 + 1 \otimes h_i.
\end{aligned} \tag{32}$$

However, it is easily checked that our coproduct Δ can be obtained from Δ_T by means of the twisting F:

$$\Delta(x) = F\Delta_T(x)F^{-1} \tag{33}$$

where

$$F = \exp\left(\hbar\sum_{i,j}\frac{1}{2}(\alpha^{-1})_{ij}H_i \otimes H_j\right). \tag{34}$$

Our antipode is obtained form the one given in [19] by means of the same twisting (see, e.g., [20]) and due to its uniqueness [21] it should coincide with (28).

Given a twisting F, the twisted R-matrix is obtained via the formula

$$R_F = F_{21}RF^{-1} \tag{35}$$

where $F_{21} = \tau \circ F$. In our case $F_{21} = F$ and F commutes with R so $R_F = R$ and we may use the formula from [19] which in terms of conformal generators reads

$$R = K\check{R}_1\check{R}_4\check{R}_3\check{R}_2 \tag{36}$$

where

$$\begin{aligned}
K &= q^{M_0\otimes M_0}q^{D\otimes D}\\
\check{R}_1 &= \exp_{q^{-1}}((q-q^{-1})M_+\otimes M_-)\\
\check{R}_2 &= \exp_{q^{-2}}(-(q-q^{-1})K_-\otimes P_+)\\
\check{R}_3 &= \exp_{q^{-1}}((q-q^{-1})K_0\otimes P_0)\\
\check{R}_4 &= \exp_{q^{-2}}(-(q-q^{-1})K_+\otimes P_-).
\end{aligned} \tag{37}$$

R satisfies the following 'reality conditions'

$$R = \tau\circ R \qquad \text{and} \qquad R^+ = R^{-1} \tag{38}$$

5. The subalgebra structure of $U_q(o(3,2))$

A linear subspace L of a Hopf algebra A is called

(a) a subalgebra if $\forall_{a,b\in L}\ ab \in L$,
(b) a Hopf subalgebra if (a) and $\forall_{a\in L}\ \Delta(a) \in L\otimes L$,
(c) a quasitriangular Hopf subalgebra if (b) and it is a quasitriangular Hopf algebra.

It can be stated that in general the group of algebra automorphisms of an $U_q(g)$ is a subgroup of that of the corresponding $U(g)$. Smaller yet is the group of Hopf algebra automorphisms of $U_q(g)$. This results in a smaller set of (Hopf)-subalgebras of $U_q(g)$ compared to $U(g)$.

In our further consideration we restrict ourselves to those Hopf-subalgebras of $U_q(Sp(4,\mathbb{C}))$ which contain at least one element of the Cartan–Weyl basis (only such subalgebras are interesting from the physical point of view). We may ask for which subspaces $\mathcal{V}$ of the vector space CW spanned by the elements of the Cartan–Weyl basis there exists a Hopf subalgebra B of $U_q(Sp(4,\mathbb{C}))$ such that $B\cap CW = \mathcal{V}$. So our problem *is to determine the set*

$$\boldsymbol{P} := \{\mathcal{V} : \mathcal{V} = B\cap CW \text{ for some Hopf-algebra} B \subset U_q(Sp(4,\mathbb{C})\}. \tag{39}$$

The answer is easy to find given the set of relations listed in section 3.

$\boldsymbol{P}$ consists of:

(i) any subspace $\mathcal{V}$ of the Cartan subalgebra of $U_q(Sp(4,\mathbb{C}))$,
(ii) any subspace $\mathcal{V}$ spanned by a subset X of the Cartan–Weyl basis satisfying

$$\begin{aligned}
&e_i \in X && \Rightarrow && h_i \in X \\
&e_{-i} \in X && \Rightarrow && h_i \in X \\
&\{e_1, e_2\} \subset X && \Rightarrow && E_3, E_4 \in X \\
&\{e_{-1}, e_{-2}\} \subset X && \Rightarrow && E_{-3}, E_{-4} \in X \\
&E_3 \in X \quad \text{or} \quad E_4 \in X && \Rightarrow && e_1, e_2, h_1, h_2 \in X \\
&E_{-3} \in X \quad \text{or} \quad E_{-4} \in X && \Rightarrow && e_{-1}, e_{-2}, h_1, h_2 \in X.
\end{aligned} \tag{40}$$

The subsets X of the Cartan–Weyl basis satisfying 2) are listed below:

$\{e_1, h_1\}, \{e_{-1}, h_1\}, \{e_2, h_2\}, \{e_{-2}, h_2\}, \{e_1, e_{-1}, h_1\}, \{e_2, e_{-2}, h_2\}, \{e_1, h_1, h_2\}$, $\{e_{-1}, h_1, h_2\}, \{e_2, h_1, h_2\}, \{e_{-2}, h_1, h_2\}, \{e_1, e_{-1}, h_1, h_2\}, \{e_2, e_{-2}, h_1, h_2\}$, $\{e_1, h_1, e_{-2}, h_2\}, \{e_2, h_2, e_{-1}, h_1\}, \{e_1, h_1, e_2, h_2, E_3, E_4\}, \{e_{-1}, h_1, e_{-2}, h_2, E_{-3}, E_{-4}\}$, $\{e_1, e_{-1}, h_1, e_2, h_2, E_3, E_4\}, \{e_1, e_{-1}, h_1, e_{-2}, h_2, E_{-3}, E_{-4}\}, \{e_1, h_1, e_2, e_{-2}, h_2, E_3, E_4\}$, $\{e_1, e_{-1}, h_1, e_2, e_{-2}, h_2, E_{-3}, E_{-4}\}, \{e_1, e_{-1}, h_1, e_2, e_{-2}, h_2, E_3, E_{-3}, E_4, E_{-4}\}$.

However, only

$$\{e_1, e_{-1}, h_1\}, \{e_2, e_{-2}, h_2\}, \{e_1, e_{-1}, h_1, h_2\}, \{e_2, e_{-2}, h_1, h_2\},$$
$$\{e_1, e_{-1}, h_1, e_2, e_{-2}, h_2, E_3, E_{-3}, E_4, E_{-4}\}$$

and all subspaces of the Cartan subalgebra generate quasitriangular Hopf algebras.

All subalgebras generated by elements of the subspaces $\mathcal{V}$ belonging to P may be treated as real subalgebras of $U_q(o(3,2))$ because they are preserved by the $^+$-involution (13).

From the physical point of view the following algebras are of special interest.

The Lorentz algebra generated by $\{M_-, M_+, M_0\} = \{e_1, e_{-1}, h_1\}$ after deformation forms a quasitriangular Hopf algebra with the R-matrix

$$R = q^{M_0 \otimes M_0} \check{R}_1 \tag{41}$$

where $\check{R}_1$ is defined in (37). The direction of the deformation is not time-like and M_0 is not the generator of shifts in time.

The Poincaré algebra generated by $\{M_+, M_-, M_0, P_+, P_-, P_0\}$ remains a subalgebra but not a Hopf subalgebra after deformation. The smallest Hopf subalgebra containing it is the Weyl algebra obtained by adding the dilatation generator D (necessary for the coproduct to close) to the set of generators.

The Weyl algebra generated by

$$\{M_+, M_-, M_0, P_+, P_-, P_0, D\} = \{e_1, e_{-1}, h_1, e_{-2}, h_2, E_{-3}, E_{-4}\}$$

becomes a Hopf algebra after deformation but not a quasitriangular one. Its R-matrix is equal to that of $U_q(o(3,2))$ and cannot be expressed in terms of the Weyl algebra generators.

6. Final remarks

We have the following chain of Hopf algebra inclusions:

$$U_q(o(2,1)) \subset U_q(\mathcal{P}_3 \oplus D) \subset U_q(o(3,2)) \tag{42}$$

as compared to

$$U_q(o(3,1)) \subset U_q(\mathcal{P}_4 \oplus D) \subset U_q(o(4,2)) \tag{43}$$

which is true in the $D = 4$ case. It should be stressed that there is an essential difference between the sequences (42) and (43): in case of $D = 4$ conformal algebra (sequence (43)) the real forms are necessarily non-standard [8, 22], of type (11). However, in both cases only the Lorentz subalgebra and the conformal algebra are quasitriangular. And in both cases in order to obtain the quantum algebra the dilatation sector must be added to the Poincaré algebra. It should be stressed that the appearance of the dilatation operator is caused by the choice of the Drinfeld–Jimbo type of deformations. It has been claimed recently [23] that by choosing other types of deformations this unpleasant feature can be avoided.

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